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Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation ☆

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Abstract

This paper discusses a randomized non-autonomous logistic equation $dN(t) = N(t)[(a(t) - b(t)N(t))\,dt + \alpha(t)\,dB(t)]$, where $B(t)$ is a 1-dimensional standard Brownian motion. In [D.Q. Jiang, N.Z. Shi, A note on non-autonomous logistic equation with random perturbation, J. Math. Anal. Appl. 303 (2005) 164–172], the authors show that $E[1/N(t)]$ has a unique positive T-periodic solution $E[1/N_p(t)]$ provided $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\,ds > 0$. We show that this equation is stochastically permanent and the solution $N_p(t)$ is globally attractive provided $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \alpha^2(t)$. By the way, the similar results of a generalized non-autonomous logistic equation with random perturbation are yielded.

Keywords: Global stability; Stochastic permanence; Randomized logistic equation; Periodic solution; Itô's formula

1. Introduction

A simple non-autonomous logistic equation, based on ordinary differential equations, is usually denoted by

$$\dot{N}(t) = N(t)\big[a(t) - b(t)N(t)\big], \tag{1.1}$$

on $t \geqslant 0$ with initial value $N(0) = N_0 > 0$, and models the population density N of a single species whose members compete among themselves for a limited amount of food and living space, where $a(t)$ is the rate of growth and $a(t)/b(t)$ is the carrying capacity at time t, both $a(t)$ and $b(t)$ are positive continuous functions. We refer the reader to May [1] for a detailed model construction. For an autonomous system (1.1), there is a stable equilibrium point of the population. Many authors have obtained a lot of interesting results about the stability of positive solutions for the

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above system (1.1) with its general case, for example, see Globalism [2]. When parameters $a(t)$ and $b(t)$ are positive T-periodic functions, Fan in [3] has proved that Eq. (1.1) has a stable positive T-periodic solution $N^*(t)$

$$1/N^*(t) = \frac{\int_t^{t+T} \exp\{\int_t^s a(\tau)\,d\tau\} b(s)\,ds}{\exp\{\int_0^T a(\tau)\,d\tau\} - 1}, \quad t \geqslant 0.$$

The existence of a stable periodic solution is of fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of the qualitative theory of differential equations. For historical background, and the basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models, we refer to the reader to, for example, the work of Fan [3], Burton [4] and the references therein. In contrast, if we now let parameters $a(t) > 0$ and $b(t) < 0$, then Eq. (1.1) has only the local solution

$$N(t) = \frac{\exp\{\int_0^t a(s)\,ds\}}{1/N_0 - \int_0^t |b(s)| \exp\{\int_0^s a(\tau)d\tau\}\,ds} \quad (0 \leqslant t < T_e),$$

which explodes to infinity at the finite time T_e, where T_e is determined by the equation

$$1/N_0 = \int_0^{T_e} \left|b(s)\right| \exp\left\{ \int_0^s a(\tau)d\tau \right\} ds.$$

To our knowledge, logistic growth model has often been used in many cases as a basic model of cell growth, fruit fly growth, some fish grown and other more particular population growth, see [1,5–9]. Especially time-dependent logistic equation with periodic coefficients is more reasonable, for example, due to seasonality. However, since in the real world the natural growth of many populations is always affected inevitably by some random disturbance, only considering the periodicity is not enough. These factors motivate us to consider the non-autonomous logistic equation with random perturbation and the natural growth rates are subject to environmental noise (cf. Mao, Marion and Renshaw [10]). It is important from the points of biological view to discover the properties of the nondeterministic system, such as stochastic permanence and global stability, and whether the presence of a such noise affects some known results. Suppose that parameter $a(t)$ is stochastically perturbed, with

$$a(t) \to a(t) + \alpha(t)\dot{B}(t),$$

where $\dot{B}(t)$ is white noise and $\alpha^2(t)$ represents the intensity of the noise. Then this environmentally perturbed system may be described by the Itô equation

$$dN(t) = N(t)\big[\big(a(t) - b(t)N(t)\big)\,dt + \alpha(t)\,dB(t)\big], \quad t \geqslant 0, \tag{1.2}$$

where $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$, $N(0) = N_0$ and N_0 is a positive number. In this paper, we assume

(H) $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and

$$\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \alpha^2(t). \tag{1.3}$$

Remark 1.1. Mao, Marion and Renshaw [10] consider the environmentally perturbed system

$$dN(t) = N(t)\big[\big(a + bN(t)\big)\,dt + \alpha N(t)\,dB(t)\big], \quad t \geqslant 0,$$

where $a, b, \alpha > 0$ with $N(0) = N_0 > 0$. No matter how small $\alpha > 0$, they show that the solution will not explode in a finite time. This result reveals the important property that the environmental noise suppresses the explosion.

For a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Arnold [11] and Freedman [12]). However, the coefficients of Eq. (1.2) do not satisfy the linear growth condition, though they are local Lipschitz continuous, so the solution of Eq. (1.2) may explode at a finite time.

Since $B(t)$ is not periodic, we cannot expect the solution $N(t)$ to Eq. (1.2) is periodic even if $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions. In fact, as far as authors know, there are few work on periodic solutions of stochastic differential equations. In [13], the authors show that $E[1/N(t)]$ has a unique positive T-periodic solution $E[1/N_p(t)]$ provided $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \alpha^2(s)]\,ds > 0$. Here, and in the sequel, "$E[f]$" shall mean the mathematical expectation of f.

Theorem A. *(See [13].) Assume that $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on* $[0, \infty)$, $a(t) > 0$ *and* $b(t) > 0$. *Then there exists a unique continuous positive solution $N(t)$ to Eq.* (1.2) *for any initial value* $N(0) = N_0 > 0$, *which is global and represented by*

$$N(t) = \frac{\exp\{\int_0^t [a(s) - \frac{\alpha^2(s)}{2}]\,ds + \alpha(s)\,dB(s)\}}{1/N_0 + \int_0^t b(s)\exp\{\int_0^s [a(\tau) - \frac{\alpha^2(\tau)}{2}]\,d\tau + \alpha(\tau)\,dB(\tau)\}\,ds}, \quad t \geqslant 0. \tag{1.4}$$

Theorem B. *(See [13].) Suppose* (H) *holds, then $E[1/N(t)]$ of Eq.* (1.2) *has a unique positive T-periodic solution*

$$E\big[1/N_p(t)\big] = \frac{\int_t^{t+T} \exp\{\int_t^s [a(\tau) - \alpha^2(\tau)]\,d\tau\} b(s)\,ds}{\exp\{\int_0^T [a(\tau) - \alpha^2(\tau)]\,d\tau\} - 1}, \quad t \geqslant 0. \tag{1.5}$$

In addition,

$$\lim_{t \to +\infty} \big\{E\big[1/N(t)\big] - E\big[1/N_p(t)\big]\big\} = 0,$$

where $N(t)$ is the solution of Eq. (1.2) *for any initial value* $N(0) = N_0 > 0$.

In a population dynamical system, the non-explosion property of the solutions, the existence and the uniqueness of the periodic solution are often not good enough but the properties of permanence and global attractivity are more desirable since they mean the long-term survival. In this paper, we show that Eq. (1.2) is stochastically persistent and the positive solution $N_p(t)$ is globally attractive. The significant contributions of this paper are therefore clear.

The remaining part of this paper is as follows. In Section 2, we yield the stochastic permanence of Eq. (1.2); in Section 3, we give the sufficient conditions for the global attractivity of the unique positive solution $N_p(t)$; in Section 4, a more general logistic system is researched by the similar strategy.

2. Stochastic permanence of Eq. (1.2)

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space on which an increasing and right continuous family $\{\mathcal{F}_t\}_{t \in [0,\infty]}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined. Let $B(t)$ be a given 1-dimensional standard Brownian motion defined on the probability space.

Lemma 2.1. *(See [13].)*

$$E\left[\exp\left\{\int_{t_0}^{t} \alpha(s)\,dB(s)\right\}\right] = \exp\left\{\frac{1}{2}\int_{t_0}^{t} \alpha^2(s)\,ds\right\}, \quad 0 \leqslant t_0 \leqslant t.$$

For convenience and simplicity in the following discussion, we always use the notations

$$f^l = \min_{t \in [0,T]} f(t), \qquad f^u = \max_{t \in [0,T]} f(t),$$

where $f(t)$ is a continuous T-periodic function.

Lemma 2.2. *Suppose* (H) *holds, then*

$$\limsup_{t \to +\infty} E\big[1/N(t)\big] \leqslant b^u/r^l := K,$$

where $N(t)$ is a solution of Eq. (1.2) *for an initial value* $N(0) = N_0 > 0$ *and* $r(t) =: a(t) - \alpha^2(t)$.

Proof. From [13], we know

$$E\big[1/N(t)\big] = \exp\left\{\int_0^t \big[\alpha^2(s) - a(s)\big]\,ds\right\}\Big/ N_0 + \int_0^t b(s)\exp\left\{\int_s^t \big[\alpha^2(\tau) - a(\tau)\big]\,d\tau\right\} ds, \quad t \geqslant 0. \tag{2.1}$$

So, we have

$$E\big[1/N(t)\big] \leqslant e^{-r^l t}/N_0 + b^u \int_0^t e^{-r^l(t-s)}\,ds \leqslant e^{-r^l t}/N_0 + b^u/r^l.$$

This completes the proof of Lemma 2.2. □

Let $p \geqslant 1$ be chosen such that

$$0 < N_0 < \frac{a^u + \frac{1}{2}(p-1)(\alpha^u)^2}{b^l}. \tag{2.2}$$

Lemma 2.3. *Let $N(t)$ be a solution of Eq.* (1.2) *with initial value $N(0) = N_0 > 0$, then*

$$E\big(N^p(t)\big) \leqslant \left[\frac{a^u + \frac{1}{2}(p-1)(\alpha^u)^2}{b^l}\right]^p := K(p). \tag{2.3}$$

Proof. We can easily know

$$\begin{aligned} dN^p(t) &= pN^{p-1}(t)\,dN(t) + \frac{1}{2}p(p-1)N^{p-2}(t)\big(dN(t)\big)^2 \\ &= pN^p(t)\big[\big(a(t) - b(t)N(t)\big)\,dt + \alpha(t)\,dB(t)\big] + \frac{1}{2}p(p-1)N^p(t)\alpha^2(t)\,dt. \end{aligned}$$

Integrating from 0 to t and taking expectations, yields

$$E\big(N^p(t)\big) - E\big(N^p(0)\big) = \int_0^t pE\big(N^p(s)\big(a(s) - b(s)N(s)\big)\big)\,ds + \int_0^t \frac{1}{2}p(p-1)\alpha^2(s)E\big(N^p(s)\big)\,ds.$$

So,

$$\begin{aligned} \frac{dE(N^p(t))}{dt} &= pE\big(N^p(t)\big(a(t) - b(t)N(t)\big)\big) + \frac{1}{2}p(p-1)\alpha^2(t)E\big(N^p(t)\big) \\ &\leqslant p\left\{\left[a(t) + \frac{1}{2}(p-1)\alpha^2(t)\right]E\big(N^p(t)\big) - b(t)\big[E\big(N^p(t)\big)\big]^{\frac{p+1}{p}}\right\} \\ &= pE\big(N^p(t)\big)\left\{\left[a(t) + \frac{1}{2}(p-1)\alpha^2(t)\right] - b(t)\big[E\big(N^p(t)\big)\big]^{\frac{1}{p}}\right\} \\ &\leqslant pE\big(N^p(t)\big)\left\{\left[a^u + \frac{1}{2}(p-1)\big(\alpha^u\big)^2\right] - b^l\big[E\big(N^p(t)\big)\big]^{\frac{1}{p}}\right\}. \end{aligned}$$

Let $y(t) = E(N^p(t))$, then we have

$$\frac{dy(t)}{dt} \leqslant py(t)\left\{\left[a^u + \frac{1}{2}(p-1)\big(\alpha^u\big)^2\right] - b^l y^{\frac{1}{p}}(t)\right\}.$$

From (2.2), we know

$$0 < b^l y^{\frac{1}{p}}(0) = b^l N(0) < a^u + \frac{1}{2}(p-1)\big(\alpha^u\big)^2,$$

then by the standard comparison theorem we know that

$$\left[E\left(N^p(t)\right)\right]^{\frac{1}{p}} = y^{\frac{1}{p}}(t) \leqslant \frac{a^u + \frac{1}{2}(p-1)(\alpha^u)^2}{b^l},$$

i.e.,

$$E\left(N^p(t)\right) \leqslant K(p).$$

This completes the proof of Lemma 2.3. □

Definition 2.1. Equation (1.2) is said to be stochastically permanent if for any $\varepsilon > 0$, there exist positive constants $\delta = \delta(\epsilon)$, $H = H(\epsilon)$ such that

$$\liminf_{t\to+\infty} P\left\{N(t) \leqslant H\right\} \geqslant 1-\epsilon, \qquad \liminf_{t\to+\infty} P\left\{N(t) \geqslant \delta\right\} \geqslant 1-\epsilon,$$

where $N(t)$ is an arbitrary solution of the equation with initial value $N(0) > 0$.

Theorem 2.1. *Suppose* (H) *holds, then Eq.* (1.2) *is stochastically permanent.*

Proof. Let $N(t)$ is an arbitrary solution of the equation with initial value $N(0) > 0$. By Lemma 2.3, we know

$$E\left(N^p(t)\right) \leqslant K(p).$$

Now, for any $\epsilon > 0$, let $H = [K(p)/\epsilon]^{1/p}$. Then by Chebyshev's inequality, we have

$$P\left\{N(t) > H\right\} \leqslant E\left(N^p(t)\right)/H^p \leqslant K(p)/H^p = \epsilon.$$

This implies

$$P\left\{N(t) \leqslant H\right\} \geqslant 1-\epsilon.$$

By Lemma 2.2, we know

$$\limsup_{t\to+\infty} E\left[1/N(t)\right] \leqslant b^u/r^l = K.$$

Now, for any $\epsilon > 0$, let $\delta = \epsilon/K$. Then

$$P\left\{N(t) < \delta\right\} = P\left\{1/N(t) > 1/\delta\right\} \leqslant \frac{E[1/N(t)]}{1/\delta} = \delta E\left[1/N(t)\right].$$

Hence,

$$\limsup_{t\to+\infty} P\left\{N(t) < \delta\right\} \leqslant \delta K = \epsilon.$$

This implies

$$\liminf_{t\to+\infty} P\left\{N(t) \geqslant \delta\right\} \geqslant 1-\epsilon.$$

This completes the proof of Theorem 2.1. □

Remark 2.1. If the intensity of the noise $\alpha^2(t)$ is suitable small, i.e.,

$$\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \alpha^2(t),$$

the stochastically logistic system (1.2) keeps the permanent property which the original ordinary differential equation owns.

3. Global attractivity of $N_p(t)$

In this section, we turn to establish sufficient criteria for the global attractivity of $N_p(t)$ described as in Theorem B. From the proof of Lemma 2.2 (see [13]), we know

$$E\big[1/N(t)\big] = \frac{1}{N_0}\exp\left(-\int_0^t r(s)\,ds\right) + \int_0^t b(s)\exp\left(-\int_s^t r(\tau)\,d\tau\right)ds,$$

$$E\big[1/N(t+T)\big] = \frac{1}{N_0}\exp\left(-\int_0^{t+T} r(s)\,ds\right) + \int_0^{t+T} b(s)\exp\left(-\int_s^{t+T} r(\tau)\,d\tau\right)ds,$$

where $r(t) = a(t) - \alpha^2(t)$.

For the solution $N_p(t)$ of Eq. (1.2) which is defined by (1.5), by Theorem B, we know $E[1/N_p(t)]$ of Eq. (1.2) is the unique positive T-periodic solution of $E[1/N(t)]$, i.e.,

$$E\big[1/N_p(t)\big] = E\big[1/N_p(t+T)\big].$$

So, we have

$$N_0 = \frac{\exp(\int_0^T r(s)\,ds) - 1}{\int_0^T b(s)\exp(\int_0^s r(\tau)\,d\tau)\,ds} := N_0^p.$$

Thus, we know $N_p(t)$ is the unique positive solution of Eq. (1.2) with initial value $N(0) = N_0^p > 0$.

Definition 3.1. Let $N(t)$ be an arbitrary solution of Eq. (1.2) with initial value $N(0) > 0$, if

$$\lim_{t\to+\infty}\big|N(t) - N_p(t)\big| = 0, \quad \text{for almost all } \omega \in \Omega,$$

then we say $N_p(t)$ is globally attractive.

Lemma 3.1. *(See [14,15].) Suppose that a stochastic process X(t) on $t \geqslant 0$ satisfies the condition*

$$E\big|X(t) - X(s)\big|^\alpha \leqslant c|t-s|^{1+\beta}, \quad 0 \leqslant s, t < \infty,$$

for some positive constants α, β and c. Then there exists a continuous modification $\tilde{X}(t)$ of $X(t)$, which has the property that for every $\gamma \in (0, \beta/\alpha)$, there is a positive random variable $h(w)$ such that

$$p\left\{\omega\colon \sup_{0<|t-s|<h(\omega),\,0\leqslant s,t<\infty} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|^\gamma} \leqslant \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder-continuous with exponent γ.

Lemma 3.2. *Let $N(t)$ be a solution of Eq.* (1.2) *with initial value $N(0) = N_0 > 0$, then almost every sample path of $N(t)$ is uniformly continuous on $t \geqslant 0$.*

Proof. We shall consider the following stochastic integral equation instead of Eq. (1.2)

$$N(t) = N_0 + \int_0^t f\big(s, N(s)\big)\,ds + \int_0^t g\big(s, N(s)\big)\,dB(s), \tag{3.1}$$

where

$$f\big(s, N(s)\big) = N(s)\big(a(s) - b(s)N(s)\big),$$
$$g\big(s, N(s)\big) = \alpha(s)N(s).$$

Then

$$\begin{aligned}
E\big(\big|f\big(s,N(s)\big)\big|^p\big) &= E\big(N^p(s)\big|a(s)-b(s)N(s)\big|^p\big) \\
&\leqslant \frac{1}{2}E\big(N^{2p}(s)\big)+\frac{1}{2}E\big[\big(a(s)-b(s)N(s)\big)^{2p}\big] \\
&\leqslant \frac{1}{2}E\big(N^{2p}(s)\big)+2^{2p-2}E\big[a^{2p}(s)+b^{2p}(s)N^{2p}(s)\big] \\
&\leqslant \frac{1}{2}K(2p)+2^{2p-2}\big[\big(a^u\big)^{2p}+\big(b^u\big)^{2p}K(2p)\big] \\
&=: F(p),
\end{aligned} \tag{3.2}$$

and

$$E\big(\big|g\big(s,N(s)\big)\big|^p\big)=E\big(\alpha^p(s)N^p(s)\big)\leqslant\big(\alpha^u\big)^pE\big(N^p(s)\big)\leqslant\big(\alpha^u\big)^pK(p)=:G(p). \tag{3.3}$$

By the moment inequality (cf. Friedman[16] or Mao [17]) for stochastic integrals (3.1), we have that for $0\leqslant t_1<t_2<\infty$ and $p>2$,

$$E\left|\int_{t_1}^{t_2} g\big(s,N(s)\big)\,dB(s)\right|^p\leqslant\big[p(p-1)/2\big]^{p/2}(t_2-t_1)^{(p-2)/2}\int_{t_1}^{t_2}E\big|g\big(s,N(s)\big)\big|^p\,ds. \tag{3.4}$$

Let $0<t_1<t_2<\infty$, $t_2-t_1\leqslant 1$, $1/p+1/q=1$, then from (3.2)–(3.4), we yield

$$\begin{aligned}
E\big|N(t_2)-N(t_1)\big|^p &\leqslant 2^{p-1}E\left(\int_{t_1}^{t_2}\big|f\big(s,N(s)\big)\big|\,ds\right)^p+2^{p-1}E\left|\int_{t_1}^{t_2}g\big(s,N(s)\big)\,dB(s)\right|^p \\
&\leqslant 2^{p-1}\left(\int_{t_1}^{t_2}1^q\,ds\right)^{p/q}E\left(\int_{t_1}^{t_2}\big|f\big(s,N(s)\big)\big|^p\,ds\right) \\
&\quad +2^{p-1}\big[p(p-1)/2\big]^{p/2}(t_2-t_1)^{p-2/2}\int_{t_1}^{t_2}E\big|g\big(s,N(s)\big)\big|^p\,ds \\
&\leqslant 2^{p-1}\left(\int_{t_1}^{t_2}1^q\,ds\right)^{p/q}E\left(\int_{t_1}^{t_2}F(p)\,ds\right)+2^{p-1}\big[p(p-1)/2\big]^{p/2}(t_2-t_1)^{p-2/2}\int_{t_1}^{t_2}G(p)\,ds \\
&= 2^{p-1}(t_2-t_1)^{(p-1)+1}F(p)+2^{p-1}\big[p(p-1)/2\big]^{p/2}(t_2-t_1)^{p/2}G(p) \\
&\leqslant 2^{p-1}(t_2-t_1)^{p/2}\big\{(t_2-t_1)^{p/2}+\big[p(p-1)/2\big]^{p/2}\big\}M(p) \\
&\leqslant 2^{p-1}(t_2-t_1)^{p/2}\big\{1+\big[p(p-1)/2\big]^{p/2}\big\}M(p),
\end{aligned}$$

where $M(p):=F(p)+G(p)$.

We see from Lemma 3.2 that almost every sample path of $N(t)$ is locally but uniformly Hölder-continuous with exponent γ for every $\gamma\in(0,p-2/2p)$ and therefore almost every sample path of $N(t)$ is uniformly continuous on $t\geqslant 0$. This completes the proof of Lemma 3.2. □

Lemma 3.3. *(See [18].) Let $f(t)$ be a nonnegative function defined on $[0,\infty)$ such that $f(t)$ is integrable on $[0,\infty)$ and is uniformly continuous on $[0,\infty)$. Then $\lim_{t\to\infty} f(t)=0$.*

Theorem 3.1. *Suppose* (H) *holds, let $N_p(t)$ be the solution of Eq.* (1.2) *defined by* (1.5)*, then $N_p(t)$ is globally attractive.*

Proof. Let $N(t)$ be an arbitrary solution of Eq. (1.2) with initial value $N(0) = N_0 > 0$. Consider a Lyapunov function $V(t)$ defined by

$$V(t) = \left|\log N(t) - \log N_p(t)\right|, \quad t \geqslant 0.$$

By Itô's formula, we have

$$\begin{aligned} d\big(\log N(t) - \log N_p(t)\big) &= \left[\frac{dN(t)}{N(t)} - \frac{(dN(t))^2}{2N^2(t)}\right] - \left[\frac{dN_p(t)}{N_p(t)} - \frac{(dN_p(t))^2}{2N_p^2(t)}\right] \\ &= -b(t)\big(N(t) - N_p(t)\big)\,dt. \end{aligned}$$

Thus, a direct calculation of the right differential $d^+V(t)$ of $V(t)$ along the solutions leads to

$$\begin{aligned} d^+V(t) &= \operatorname{sgn}\big(N(t) - N_p(t)\big) d\big(\log N(t) - \log N_p(t)\big) \\ &= -\operatorname{sgn}\big(N(t) - N_p(t)\big)\big[b(t)\big(N(t) - N_p(t)\big)\big]\,dt \\ &= -b(t)\left|N(t) - N_p(t)\right|dt \\ &\leqslant -b^l\left|N(t) - N_p(t)\right|dt. \end{aligned} \tag{3.5}$$

Integrating (3.5) from 0 to t, we have

$$V(t) + b^l \int_0^t \left|N(s) - N_p(s)\right| ds \leqslant V(0) < \infty,$$

which leads to

$$\left|N(t) - N_p(t)\right| \in L^1[0, \infty).$$

Therefore from Lemmas 3.3 and 3.4, one obtains

$$\lim_{t\to+\infty}\left|N(t) - N_p(t)\right| = 0, \quad \text{for almost all } \omega \in \Omega.$$

This completes the proof of Theorem 3.1. □

Remark 3.1. If the intensity of the noise $\alpha^2(t)$ is suitable small, i.e.,

$$\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \alpha^2(t),$$

the stochastically logistic system (1.2) owns a globally attractive and unique solution $N_p(t)$ defined by (1.5). In addition, $E[1/N_p(t)]$ is T-periodic. This result is similar with that of the original ordinary differential equation.

From the above results, we can see that when the intensity of the noise $\alpha^2(t)$ is not too big, the presence of a such noise, essentially, does not affect some main properties of original ordinary differential equation, such as permanence, global attractivity of periodic solutions and the non-explosion property. By the way, if $b(t) < 0$, no matter how big $\alpha^2(t)$, the solution of Eq. (1.2) will explode in a finite time. This result reveals the important property that the environmental noise cannot suppress the explosion, see [13] for details.

4. Generalized results

A more general non-autonomous logistic equation, based on ordinary differential equations, is usually denoted by

$$\dot{N}(t) = N(t)\big[a(t) - b(t)N^\theta(t)\big] \quad (\theta > 0). \tag{4.1}$$

Some detailed studies about the model may be found in Gilpin and Ayala [19,20]. The authors in [13] considered the randomized model (4.2) based on (4.1) with intensity $\alpha^2(t)$

$$dN(t) = N(t)\big[\big(a(t) - b(t)N^\theta(t)\big)\,dt + \alpha(t)\,dB(t)\big], \quad t \geqslant 0, \tag{4.2}$$

where $\theta > 0$ is an odd number, $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$, $N(0) = N_0$ and N_0 is a positive random variable. Here $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$, $b(t) > 0$ and N_0 is independent of $B(t)$.

And in [13], the following results have been obtained:

Theorem C. *(See [13].) Assume that $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$ and $b(t) > 0$. Then there exists a unique continuous solution $N(t)$ to Eq.* (4.2) *for any initial value $N(0) = N_0 > 0$, which is global and represented by*

$$N^\theta(t) = \frac{\exp\{\theta(\int_0^t [a(s) - \frac{\alpha^2(s)}{2}]\,ds + \alpha(s)\,dB(s))\}}{1/N_0^\theta + \theta \int_0^t b(s) \exp\{\theta(\int_0^s [a(\tau) - \frac{\alpha^2(\tau)}{2}]\,d\tau + \alpha(\tau)\,dB(\tau))\}\,ds}, \quad t \geqslant 0.$$

Theorem D. *(See [13].) Suppose $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and $\int_0^T [a(s) - \frac{\theta+1}{2}\alpha^2(s)]\,ds > 0$. Then $E[1/N^\theta(t)]$ of Eq.* (4.2) *has a unique positive T-periodic solution $E[1/N_p^\theta(t)]$ which is represented by*

$$E\big[1/N_p^\theta(t)\big] = \frac{\theta \int_t^{t+T} \exp\{\int_t^s [\theta a(\tau) - \frac{\theta(\theta+1)}{2}\alpha^2(\tau)]\,d\tau\} b(s)\,ds}{\exp\{\int_0^T [\theta a(\tau) - \frac{\theta(\theta+1)}{2}\alpha^2(\tau)]\,d\tau\} - 1}, \quad t \geqslant 0. \tag{4.3}$$

In addition,

$$\lim_{t\to+\infty} \big\{E\big[1/N^\theta(t)\big] - E\big[1/N_p^\theta(t)\big]\big\} = 0,$$

where $N(t)$ is the solution of Eq. (4.2) *for any initial value $N(0) = N_0 > 0$.*

Let $N(t)$ be a solution of Eq. (4.2), by Itô's formula

$$dN^\theta(t) = N^\theta(t)\bigg[\bigg(\theta a(t) + \frac{\theta(\theta-1)}{2}\alpha^2(t) - \theta b(t) N^\theta(t)\bigg)\,dt + \theta\alpha(t)\,dB(t)\bigg]. \tag{4.4}$$

We assume

$(\mathrm{H})_\theta$ $a(t)$, $b(t)$ and $\alpha(t)$ are continuous T-periodic functions, $a(t) > 0$, $b(t) > 0$ and

$$\min_{t\in[0,T]} a(t) > \max_{t\in[0,T]} \frac{\theta+1}{2}\alpha^2(t). \tag{4.5}$$

Similarly to the proof of Theorems 2.1 and 3.1, we have the following results.

Theorem 4.1. *Suppose* $(\mathrm{H})_\theta$ *holds, then Eq.* (4.2) *is stochastically permanent.*

Theorem 4.2. *Suppose* $(\mathrm{H})_\theta$ *holds, let $N_p(t)$ be the solution of Eq.* (4.2) *defined by* (4.3)*, then*

$$\lim_{t\to+\infty} \big\{N^\theta(t) - N_p^\theta(t)\big\} = 0,$$

where $N(t)$ is an arbitrary solution of Eq. (4.2) *with initial value $N(0) > 0$.*

Remark 4.1. Theorems 4.1 and 4.2 generalize the main results of randomized non-autonomous logistic equation (1.2), i.e. Theorems 2.1 and 3.1.

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